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Thermodynamics of a two-dimensional self-gravitating system

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The mean field thermodynamics of a system of N gravitationally interacting particles confined in some bounded plane domain Ω is considered in the four possible situations corresponding to the following two pairs of alternatives: (a) Confinement is due either to a rigid circular wall $\partial\Omega$ or to an imposed external pressure (in which case $\partial\Omega$ is a free boundary). (b) The system is either in contact with a thermal bath at temperature T, or it is thermally insulated. It is shown in particular that (i) for a system at given temperature T, a globally stable equilibrium (minimum free energy or minimum free enthalpy state for $\partial\Omega$ rigid or free, respectively) exists and is unique if and only if T exceeds a critical value T_c, and (ii) for a thermally insulated system, a unique globally stable (maximum entropy) equilibrium exists for any value of the energy (rigid $\partial\Omega$) or of the enthalpy (free $\partial\Omega$). The case of a system confined in a domain of arbitrary shape is also discussed. Bounds on the free energy and the entropy are derived, and it is proven that no isothermal equilibrium (stable or unstable) with a temperature $T \le T_c$ can exist if the domain is "star shaped."

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I. INTRODUCTION

One-dimensional (1D) and two-dimensional self-gravitating systems are interesting "toy models" which are quite useful for testing our ideas about the statistical mechanics of systems governed by long-range attractive interaction. They have been yet the object of several investigations, concerning in particular the existence of equilibrium states, and the possible relaxation to an equilibrium.

As for the existence of equilibrium states, a few results have been obtained in the framework of "exact" equilibrium statistical mechanics (in which one considers a finite number N of particles). For instance, it has been shown [1,2] that, for a system confined inside a box, a microcanonical description (at any energy) and a canonical one (but only above a critical temperature in 2D) are possible without the need of introducing stabilizing cutoff (this is in contrast with the 3D case; see, e.g., Ref. [2] for a review), and the exact equations of state have been derived. For an unconfined 1D system with a fixed center of mass, both descriptions have also been proven to hold. Moreover, the associated one-particle distribution functions $f_N^{(1)}$ have been computed in closed form [3], and the spatial correlations between a pair of particles have been evaluated [4].

An important point which emerges from these studies (and which is physically quite obvious) is the nonextensivity of the usual thermodynamic functions (energy, etc.), which precludes the existence when $N \to \infty$ of the *standard thermodynamic limit* (i.e., the infinite volume limit at constant density and energy per particle). But simple scaling considerations suggest the existence of an *inhomogeneous mean field thermodynamic limit* when $N \to \infty$ at constant volume under imposed appropriate scalings—in the canonical ensemble, for instance, the temperature may be taken to increase like N (to compensate for the increase in the strength of the interaction of one particle with the other ones), while the energy scales like N^2. This view can be checked to be correct for 1D unconfined systems [3,4]. In that case, it can be shown explicitly indeed that both the microcanonical and canonical $f_N^{(1)}$ approach at large N a stationary Maxwellian solution to Vlasov's equation, as expected *a priori* on intuitive grounds. For 2D systems, the calculation of $f_N^{(1)}$ has not been achieved, and an exact direct determination of a mean field state is not possible. The existence of the latter in the canonical ensemble, however, has been rigorously proven most recently by Kiessling [5] and by Caglioti *et al.* [6], who extended to the case of singular logarithmic interactions some earlier work by Messer and Spohn [7]. They showed in particular that, above a critical temperature T_c (below which the partition function is not defined, see above), the one particle function converges to the Maxwellian solution f to Vlasov's equation which minimizes *globally* the mean field Helmholtz's free energy (or to a superposition of such solutions, if several do exist). Actually, the mean field equation satisfied by f was considered earlier—taking its validity for granted—by Katz and Lynden-Bell [8] in their phenomenological study of a system enclosed in a circular domain and maintained in contact with a heat bath at temperature T. They were able to prove (however, by restricting their attention to the only cylindrically symmetric functions f) that an equilibrium state minimizing locally the Helmholtz's free energy exists (and is unique) when $T > T_c$, the value of the critical temperature thus emerging also naturally from this simple macroscopic approach. Some considerations on the mean field limit have also been reported in Refs. [9,10].

The problem of the approach towards equilibrium has been considered by many authors for 1D systems from both the analytical [11] and the numerical points of view [12]. All these studies seem actually to indicate that there may be in general some degree of persistence of the correlations when a 1D system evolves and that consequently relaxation to equilibrium is possibly somewhat incomplete. The case of 2D systems has not yet been considered in details owing to numerical difficulties. But it can be hoped that precise 2D codes will become available in the near future, and, for comparison purposes, it will be certainly quite interesting at that time to know the equilibrium states to which a system may possibly relax.

It is the aim of this paper to develop in the framework of the phenomenological mean field approximation a calculation of these states, thus assumed to be describable by a one-particle distribution function which is a solution to Vlasov's equation globally minimizing (or maximizing) the thermodynamic potential appropriate to the externally imposed constraints. As for the latter, we shall consider here systems which are either in contact with a heat bath or thermally insulated, and are confined either by an external pressure or by a circular boundary (we shall also discuss the case of a boundary of arbitrary shape, but only briefly to avoid going into too much mathematical details). Note that for a box-confined system at a given temperature (a case also investigated in Ref. [5], but by methods different from ours), we are guaranteed by the recent results described above that a state minimizing globally the free energy (the appropriate potential here) is, at least if it is unique, an *exact asymptotic mean field state*. For the three other situations, there are not yet mathematical results characterizing rigorously the mean field states, but it seems quite likely that the states we shall determine, insofar as they are unique, are also asymptotically exact.

It is worth noticing that the problem considered here has some strong formal connections with that of the 2D statistical equilibrium of either an ideal fluid [6] or of a current-carrying magnetized plasma [13] (this last concept has been introduced in Ref. [14] and is useful in both laboratory and cosmical contexts). In all these cases indeed, one is led to a so-called Vlasov-Poisson-Emden equation for a potential (see Ref. [15] for a review of some of the properties of this equation). There is an important difference, however, in the nature of the boundary conditions which have to be imposed on the relevant potential. For gravitational problems, we fix a condition only at infinity (physically, there is no gravitational screens, and a box can only confine particles, not interactions), while in the other problems referred to above, the potential is prescribed to take given values on the boundary (in the magnetostatic problem, for instance, this corresponds to the fact that the magnetic flux distribution is kept fixed by the conducting wall).

The paper is organized as follows. In Sec. II, we list the assumptions of the model and give a precise statement of the problems we want to consider. In Sec. III, we introduce two functional transforms which will be of repeated use hereafter. In Secs. IV and V, we consider a box-confined system in contact with a thermal bath and thermally insulated, respectively. The case of a pressure-confined system (in the two previous thermal situations) is treated in Sec. VI. Results are summarized and compared with those known for a 3D system in the concluding section, Sec. VII.

II. STATEMENT OF THE PROBLEM

A. Assumptions

We are interested in a plane system constituted of N particles of mass m interacting through their gravitational field (equivalently, we can think of the system as being made of N homogeneous parallel rods embedded in the 3D space $\mathbb{R}_r^3$). For convenience, we shall work here only with dimensionless variables—masses, lengths, velocities, energies, temperatures, and pressures being normalized to m, L_0, $V_0:=(Gm)^{1/2}$, $E_0:=mV_0^2=Gm^2$, $T_0:=mV_0^2k^{-1}$, and $p_0:=kT_0L_0^{-2}$, respectively, where L_0 is some arbitrarily chosen quantity, G is the gravitational constant appropriate to the model (G is dimensionally related to the usual Newtonian constant G_N by $[G]=[G_N][L]^{-1}$), and k is the Boltzmann constant.

We assume the following.

(a) In the plane $\mathbb{R}_r^2:=\{\mathbf{r}\}$, the system is confined either by a rigid regular curve $\partial\Omega$ limiting a bounded connected domain Ω or by a given external constant pressure P.

(b) The system is either maintained in contact with a thermal bath at temperature T or it is thermally insulated.

(c) A state of the system is completely characterized by a single-particle distribution function $f(\mathbf{r},\mathbf{v})$ ($\mathbf{v}$ is the velocity of a particle) defined on the phase space $\Gamma:=\mathbb{R}_r^2\times\mathbb{R}_v^2=\{\xi:=(\mathbf{r},\mathbf{v})\}$. Of course, when the system is pressure confined, f must belong to the set

$$\mathcal{G}_0[N]:=\{f\,|\,f\geq 0\ ;\mathrm{diam}\Omega_f<\infty\ ;\int f(\xi)d\xi=N\}\ , \quad (2.1)$$

where $\Omega_f\subset\mathbb{R}_r^2$ denotes the support of the density of particles

$$n_f(\mathbf{r})=\int f(\xi)d\mathbf{v} \quad (2.2)$$

associated with f and "diam" stands for "diameter" [unless otherwise specified, all the integrals with respect to $d\xi$ ($d\mathbf{r}$ and $d\mathbf{v}$, respectively) are taken over the whole space Γ ($\mathbb{R}_r^2$ and $\mathbb{R}_v^2$, respectively)]. For a system confined inside the rigid box Ω, f is an element of

$$\mathcal{F}_0[\Omega,N]:=\{f\,|\,f\in\mathcal{G}_0[N]\ ,\ \ \Omega_f\subset\bar{\Omega}\}\ . \quad (2.3)$$

(d) Interaction between the particles is mediated by the mean potential

$$\phi_f(\mathbf{r}):=-2\int f(\xi')\ln\frac{1}{|\mathbf{r}-\mathbf{r}'|}d\xi' = -2\int n_f(\mathbf{r}')\ln\frac{1}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'\ , \quad (2.4)$$

which satisfies the usual Poisson equation and asymptotic condition

$$\Delta\phi_f=4\pi n_f \quad \text{in } \mathbb{R}_r^2\ , \quad (2.5a)$$

$$\lim_{r\to\infty}\{\phi_f(\mathbf{r})-2N\ln r\}=0 \quad (2.5b)$$

($r:=|\mathbf{r}|$). Note that our definition (2.4) contains an implicit choice of gauge (we have assumed arbitrarily that the potential created by a particle vanishes at a unit distance from it). Some properties of the potential ϕ_f are recalled in Appendix A.

B. Definitions

Let us now define a series of functionals of f: (a) kinetic energy ($v := |\mathbf{v}|$)

$$E_c[f] := \frac{1}{2}\int v^2 f(\xi) d\xi \ ; \quad (2.6)$$

(b) potential energy

$$E_p[f] := -\int f(\xi) f(\xi') \ln \frac{1}{|\mathbf{r}-\mathbf{r}'|} d\xi\, d\xi' = -\int n_f(\mathbf{r}) n_f(\mathbf{r}') \ln \frac{1}{|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}\, d\mathbf{r}' \ ; \quad (2.7)$$

(c) total energy

$$E[f] := E_c[f] + E_p[f] \ ; \quad (2.8)$$

(d) entropy

$$S[f] := -\int f(\xi) \ln f(\xi) d\xi \ ; \quad (2.9)$$

(e) free energy for the system in contact with a heat bath at temperature T,

$$F[T,f] := E[f] - TS[f] \ ; \quad (2.10)$$

(f) enthalpy for the system confined by the external pressure P,

$$H[P,f] := E[f] + P|\Omega_f| \ , \quad (2.11)$$

where $|A|$ denotes quite generally the measure of the subset $A \subset \mathbb{R}^2_r$; and (g) free enthalpy for the system confined by the external pressure P and in contact with a thermal bath at temperature T,

$$G[P,T,f] := E[f] - TS[f] + P|\Omega_f| \ . \quad (2.12)$$

We shall denote by $\mathcal{F}[\Omega,N]$, or $\mathcal{F}$ for short ($\mathcal{G}[N]$ or $\mathcal{G}$, respectively), the subset of $\mathcal{F}_0[\Omega,N]$ ($\mathcal{G}_0[N]$) containing the distribution functions having finite kinetic energy, potential energy, and entropy. Moreover, to remove an essential degeneracy existing in the case of a pressure confined system, for which states corresponding to each other by spatial translations are equivalent, we impose the f in $\mathcal{G}$ to have their center of mass coinciding with the origin O of $\mathbb{R}^2_r$. Also, we shall denote by $\mathcal{F}[\Omega,N,E]$, or $\mathcal{F}[E]$ ($\mathcal{G}[N,H]$ or $\mathcal{G}[H]$, respectively), the subset of $\mathcal{F}$ ($\mathcal{G}$) containing the distribution functions having the prescribed value E (H) of the energy (enthalpy). Two points are worth noticing here.

(i) These sets of functions are never empty. Let indeed Ω be a fixed confining domain, D_0 be a disk of radius R_0 contained in Ω, and consider the distribution function $f_{RV} := N(\pi^2 R^2 V^2)^{-1}\, \theta(R-r)\, \theta(V-v)$, where R ($0<R<R_0$) and V ($0<V$) are arbitrary parameters, θ is the usual step function, and the origin O of $\mathbf{r}$ has been chosen at the center of D_0. Then it is easy to check that f_{RV} belongs to $\mathcal{F}$ and $\mathcal{G}$ and that R and V can be chosen in such a way that f_{RV} belongs either to $\mathcal{F}[E]$ for any fixed value of E or to $\mathcal{G}[H]$ for any fixed value of H ($E[f_{RV}] = NV^2/4 + N^2(\ln R - \frac{1}{4})$ can be made indeed to vary continuously between $-\infty$ ($R \to 0$) and $+\infty$ ($V \to +\infty$)).

(ii) For any function f of $\mathcal{F}$ or $\mathcal{G}$, we have

$$N \ln \left| \frac{N}{|\Omega_f|} \right| \le \int n_f \ln n_f d\mathbf{r} \le \int |n_f \ln n_f| d\mathbf{r} \le 2\pi E_c[f] - S[f] + \frac{2|\Omega_f|}{e} \quad (2.13)$$

by Jensen's inequality associated with the convex function $x \ln x$ and the probability measures $d\mathbf{r}|\Omega_f|^{-1}$ (first inequality) and $e^{-\pi v^2} d\mathbf{v}$ (last inequality), respectively. Therefore the function of compact support $n_f \ln n_f$ is in $L^1(\mathbb{R}^2_r)$, which implies [6] that the associated potential ϕ_f is continuous and thus bounded on Ω_f (note that the condition $|E_p[f]| < \infty$ follows from that result and could thus be forgotten in the definition of $\mathcal{F}$ and $\mathcal{G}$).

C. Problems

Our aim in this paper is to address the following questions: Is it possible for the system to settle down to an absolutely stable thermodynamic equilibrium in the following situations: (a) confinement by a box and thermal contact with a thermostat—the system is seeking for a state minimizing the free energy $F[T,\ ]$ in $\mathcal{F}$; (b) confinement by box and thermal insulation—energy is kept fixed at its initial value E during any relaxation process and the system tries to reach a state maximizing the entropy $S[\ ]$ in $\mathcal{F}[E]$; (c) confinement by an external pressure and contact with a thermal bath—the system looks for a state minimizing the free enthalpy $G[P,T,\ ]$ in $\mathcal{G}$; and (d) confinement by an external pressure and thermal insulation—the system evolves at constant enthalpy H and is seeking a state maximizing the entropy $S[\ ]$ in $\mathcal{G}[H]$.

Of course, these problems are strongly related to each other, as it is evident *a priori* and will be shown in details hereafter.

III. TWO USEFUL STATE TRANSFORMS

We introduce in this section two functional transforms which will be the key tools for deriving our main results below.

A. Symmetrization of a distribution function

To any function f belonging to $\mathcal{F}$, we associate its "decreasing spherical rearrangement" with respect to $\mathbf{r}$, denoted as f^* [16]. For $\mathbf{v}$ fixed, $f^*(r,\mathbf{v})$ is the essentially unique cylindrically symmetric function defined on $\mathbb{R}^2_r$ which is nonincreasing with $r = |\mathbf{r}|$ and which satisfies $|\{\mathbf{r}|f(\mathbf{r},\mathbf{v})\rangle\tau\}| = |\{\mathbf{r}|f^*(r,\mathbf{v})\rangle\tau\}|$ for any $\tau \ge 0$. As it is well known, the symmetrization mapping $f \to f^*$ conserves the values of all the functionals of the form $\int X(f,\mathbf{v}) d\xi$. Then, in particular, f^* belongs to $\mathcal{F}^* := \mathcal{F}[\Omega^*,N]$, where Ω^* is the disk of center 0 and radius $R^* = (|\Omega|\pi^{-1})^{1/2}$ ($|\Omega| = |\Omega^*| = \pi R^{*2}$) and

$$E_c[f^*] = E_c[f] \ , \quad (3.1)$$

$$S[f^*] = S[f] \ . \quad (3.2)$$

The potential energy of f, on the contrary, is decreased if $f\neq f^*$:

$$E_p[f^*]\leq E_p[f] \tag{3.3}$$

(see Appendix B). Whence, with the help of Eqs. (3.1) and (3.2),

$$E[f^*]\leq E[f] , \tag{3.4}$$

$$F[T,f^*]\leq F[T,f] , \tag{3.5}$$

with equality holding in both relations if and only if $\Omega=\Omega^*$ and $f=f^*$.

B. Maxwell-Boltzmann distribution functions

We also associate with any f of $\mathcal{F}$ the "Maxwell-Boltzmann" distribution function

$$\tilde{f}_\lambda(\mathbf{r},v):=\frac{\lambda N}{2\pi}\frac{e^{-\lambda[v^2/2+\phi_f(\mathbf{r})]}}{\int_\Omega e^{-\lambda\phi_f(\mathbf{r}')}d\mathbf{r}'}\chi_\Omega(\mathbf{r}) , \tag{3.6}$$

where $\lambda>0$ is an arbitrary parameter (λ^{-1} will be called the temperature of $\tilde{f}_\lambda$) and χ_A denotes quite generally the characteristic function of $A\subset\mathbb{R}_r^2$. Owing to the property of the Newtonian potential recalled in Sec. II B, it is clear that $\tilde{f}_\lambda$ is well defined and belongs to $\mathcal{F}$ too; also $n_{\tilde{f}_\lambda}$ is bounded, and then $\phi_{\tilde{f}_\lambda}$ is continuously differentiable in $\mathbb{R}_r^2$.

A function f is kept invariant by the transform $f\rightarrow\tilde{f}_\lambda$ if and only if it is of the form

$$f(\mathbf{r},v)=\frac{\lambda N}{2\pi}\frac{e^{-\lambda[v^2/2+\phi_f(\mathbf{r})]}}{\int_\Omega e^{-\lambda\phi_f(\mathbf{r}')}d\mathbf{r}'}\chi_\Omega(\mathbf{r}) , \tag{3.7}$$

its potential ϕ_f thus solving the nonlinear integral equation

$$\phi(\mathbf{r})=-2N\left[\int_\Omega e^{-\lambda\phi(\mathbf{r}')}\ln\frac{1}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'\right]\times\left[\int_\Omega e^{-\lambda\phi(\mathbf{r}')}d\mathbf{r}'\right]^{-1} , \tag{3.8}$$

or equivalently the nonlinear Poisson equation

$$\Delta\phi(\mathbf{r})=4\pi N\frac{e^{-\lambda\phi(\mathbf{r})}}{\int_\Omega e^{-\lambda\phi(\mathbf{r}')}d\mathbf{r}'}\chi_\Omega(\mathbf{r}) \tag{3.9}$$

with the asymptotic condition (2.5b).

A simple calculation [using in particular the reciprocity and positivity relations (A5) and (A6)] gives

$$E_c[\tilde{f}_\lambda]=N\lambda^{-1} , \tag{3.10}$$

$$\begin{aligned}E[\tilde{f}_\lambda]-E[f]&=\int(\tilde{f}_\lambda-f)\left[\frac{v^2}{2}+\phi_f\right]d\xi\\&\quad+\tfrac{1}{2}\int(\tilde{f}_\lambda-f)(\phi_{\tilde{f}_\lambda}-\phi_f)d\xi\\&\leq\int(\tilde{f}_\lambda-f)\left[\frac{v^2}{2}+\phi_f\right]d\xi ,\end{aligned} \tag{3.11}$$

with equality if and only if $n_f=n_{\tilde{f}_\lambda}$. On the other hand, by applying the standard inequality (resulting from the convexity of the function $x\ln x$) $y\ln y-x\ln x>(y-x)(1+\ln x)$ $(x,y>0;\, x\neq y)$, we obtain

$$\begin{aligned}S[\tilde{f}_\lambda]-S[f]&\geq-\int(\tilde{f}_\lambda-f)\ln\tilde{f}_\lambda d\xi\\&=\lambda\int(\tilde{f}_\lambda-f)\left[\frac{v^2}{2}+\phi_f\right]d\xi ,\end{aligned} \tag{3.12}$$

with equality if an only if $f=\tilde{f}_\lambda$.

Equations (3.11) and (3.12) imply in particular

$$\begin{aligned}F[\lambda^{-1},\tilde{f}_\lambda]-F[\lambda^{-1},f]&=\left\{\frac{N}{\lambda}\ln\left[\frac{N\lambda}{2\pi}\right]-\frac{N}{\lambda}\ln\left[\int_\Omega e^{-\lambda\phi_f}d\mathbf{r}\right]-E_p[f]\right\}+\tfrac{1}{2}\int(\tilde{f}_\lambda-f)(\phi_{\tilde{f}_\lambda}-\phi_f)d\mathbf{r}-F[\lambda^{-1},f]\\&\leq\tfrac{1}{2}\int(\tilde{f}_\lambda-f)(\phi_{\tilde{f}_\lambda}-\phi_f)d\xi\leq0 ,\end{aligned} \tag{3.13}$$

where we have used once more Eqs. (A5) and (A6), and equality holds if and only if $f=\tilde{f}_\lambda$. Therefore, the transform $f\rightarrow\tilde{f}_\lambda$ decreases the free energy corresponding to the temperature λ^{-1}, but if $f=\tilde{f}_\lambda$, i.e., if Eqs. (3.7)–(3.9) are satisfied by f and ϕ_f.

To conclude this section, we quote two important properties of the functions $\tilde{f}_\lambda$ associated with a given f of $\mathcal{F}$.

(a) $S(\lambda):=S[\tilde{f}_\lambda]$ increases monotonically from $-\infty$ to $+\infty$ when λ increases from 0 to $+\infty$. We have indeed $\lim_{\lambda\rightarrow0}S(\lambda)=+\infty$ (obvious), $\lim_{\lambda\rightarrow\infty}S(\lambda)=-\infty$ [by Jensen's inequality $S(\lambda)\leq N\ln(2\pi e|\Omega|/N\lambda)$], and $S'(\lambda)<-\lambda^{-1}<0$ (by Schwartz's inequality). Then there is always a unique value $\mu[f]$ of λ such that $S[f]=S[\tilde{f}_\mu]$.

(b) There is always a value $\nu[f]$ of λ ($\mu[f]\leq\nu[f]<\infty$) such that $E[\tilde{f}_\nu]=E[f]$. We have indeed $\lim_{\lambda\rightarrow\infty}E[\tilde{f}_\lambda]=+\infty>E[f]$ and $E[\tilde{f}_\mu]\leq E[f]$ by using the previous result and Eqs. (3.11) and (3.12).

IV. SYSTEM CONFINED BY A BOX AND IN CONTACT WITH A THERMAL BATH

In this section, we assume that the system is contained in a rigid box Ω and is maintained in contact with a thermal bath at temperature $T=:\beta^{-1}$. We consider in details only the case where Ω is a disk of radius R centered at the origin, contenting ourselves of making a few points about the general case.

A. General statements and lower bound on the free energy (Ω a disk)

Because of the free energy decreasing character of symmetrization (which in the case where Ω is a disk is an internal mapping of $\mathcal{F}$), it is clear that we may restrict our search for the minimizers of F to the subset $\mathcal{F}_s$ of $\mathcal{F}$ containing all the functions which are cylindrically symmetric in $\mathbb{R}^2_r$. An important point is that the potential of any function f of $\mathcal{F}_s$ satisfies necessarily the boundary conditions (Appendix A)

$$\phi_f(R)=2N\ln R \ , \tag{4.1}$$

$$\frac{d\phi_f}{dr}(R)=\frac{2N}{R} \ , \tag{4.2}$$

and it is possible by using Gauss's theorem to write

$$E_p[f]=N^2\ln R-\frac{1}{8\pi}\int_\Omega |\nabla\phi_f|^2 d\mathbf{r} \ . \tag{4.3}$$

For f in $\mathcal{F}_s$, ϕ_f thus belongs to the set of potentials on Ω

$$\mathcal{P}_s[\Omega,N]:=\{\phi|\phi(\mathbf{r})=\phi(r);\phi(R)=2N\ln R;\int_\Omega|\nabla\phi|^2d\mathbf{r}<\infty\} \ . \tag{4.4}$$

We note that, by the so-called Moser-Trudinger inequality [Eq. (B6)], we have for an arbitrary ϕ belonging to $\mathcal{P}_s$

$$\int_\Omega e^{-\beta\phi}d\mathbf{r}=e^{-\beta\phi(R)}\int_\Omega e^{-\beta[\phi-\phi(R)]}d\mathbf{r} \le \frac{c|\Omega|}{R^{2\beta N}}\exp\left[\frac{\beta^2}{16\pi}\int_\Omega|\nabla\phi|^2d\mathbf{r}\right] , \tag{4.5}$$

with c an absolute constant (independent of Ω).

Let us now associate with any f of $\mathcal{F}_s$, the Maxwell-Boltzmann function $\tilde{f}_\beta$, which of course belongs to $\mathcal{F}_s$ too. Using Eqs. (3.13) and (4.3), and Gauss's theorem, we can write

$$F[T,\tilde{f}_\beta]=J[T,\phi_f]-\frac{1}{8\pi}\int_\Omega|\nabla(\phi_f-\phi_{\tilde{f}_\beta})|^2d\mathbf{r} \le J[T,\phi_f]\le F[T,f] \ , \tag{4.6}$$

with equality holding if and only if $f=\tilde{f}_\beta$; $J[T,\phi]$ denotes here the functional

$$J[\Omega,T,N,\phi]:=\frac{1}{8\pi}\int_\Omega|\nabla\phi|^2d\mathbf{r}-NT\ln\left[\int_\Omega e^{-\beta\phi}d\mathbf{r}\right] +NT\ln\left[\frac{N\beta}{2\pi R^{\beta N}}\right] , \tag{4.7}$$

which is well defined on $\mathcal{P}_s$. By Eq. (4.5), we have for any ϕ in $\mathcal{P}_s$

$$J[T,\phi]\ge NT\ln\left[\frac{\beta N}{2\pi^2 c}\left[\frac{\pi}{|\Omega|}\right]^{1-\beta N/2}\right] +\frac{1}{8\pi}\left[1-\frac{\beta N}{2}\right]\int_\Omega|\nabla\phi|^2d\mathbf{r} \ . \tag{4.8}$$

Therefore, if

$$T\ge T_c:=N/2 \quad (\beta N\le 2) \ , \tag{4.9}$$

J is bounded from below on $\mathcal{P}_s$, which implies by Eq. (4.6) that F is bounded from below on $\mathcal{F}_s$ and thus on $\mathcal{F}$.

When $T<T_c$ ($\beta N>2$), on the contrary, it turns out that F can be made as negative as we want by choosing an appropriate function in $\mathcal{F}$. Let indeed f_1 be some arbitrary element of $\mathcal{F}$, and set for any $\alpha\ge 1$

$$f_\alpha(\mathbf{r},\mathbf{v})=\alpha^2 f_1(\alpha\mathbf{r},\mathbf{v}) \ . \tag{4.10}$$

It is readily checked that f_α belongs to $\mathcal{F}$ too and that

$$F[T,f_\alpha]=F[T,f_1]+NT(2-\beta N)\ln\alpha \ , \tag{4.11}$$

whence $\lim_{\alpha\to\infty}F[T,f_\alpha]=-\infty$, i.e., the free energy can be made to decrease as much as we want by concentrating the system indefinitely.

B. Existence and uniqueness of a free energy minimizer when $T>T_c$ (Ω a disk)

Owing to the result of Sec. III B and the symmetry considerations of the preceding subsection, it is clear that if F admits a minimizer f^- in $\mathcal{F}$, it must be of the form (3.7) ($\tilde{f}_\beta^-=f^-$), with $\phi^-:=\phi_{f^-}$ being a solution in $\mathcal{P}_s$ of Eq. (3.9) (and thus a smooth potential) [17]. But it is easily checked by direct integration [6,13] that this equation has no solution for $T\le T_c$ and has the unique solution

$$\phi^-(r)=2N\ln R+\frac{2}{\beta}\ln\left[\frac{2-\beta N}{2}+\frac{\beta N}{2}\frac{r^2}{R^2}\right] \quad \text{in } \Omega \tag{4.12}$$

for $T>T_c$. Moreover, ϕ^- is the unique minimizer of $J[T,F]$ in $\mathcal{P}_s$. This results at once from the fact that Eq. (3.9) is the Euler-Lagrange equation associated with the minimization problem for J, this latter being known otherwise [6,13] to admit a solution in $\mathcal{P}_s$ (see Appendix B 3 for a proof).

Therefore (a) F does not admit any minimizer in $\mathcal{F}$ for $T\le T_c$ [this result is also a consequence of Eq. (4.11), but only for $T<T_c$] and (b) for $T>T_c$, there is a unique free energy minimizer given by

$$f^-(r,v):=\frac{N\beta}{\pi^2R^2}(2-\beta N)\frac{1}{[(2-\beta N)+\beta N r^2/R^2]^2} \times e^{-\beta v^2/2}\chi_\Omega(\mathbf{r}) \ , \tag{4.13}$$

a solution already given in Refs. [8,18]. Indeed, a minimizer of F is necessarily of this form; and f^- is actually a minimizer, as in the opposite case, there would exist in $\mathcal{F}_s$ a function f such that [by Eq. (4.6)]

$$J[T,\phi_f]\le F[T,f]<F[T,f^-]=J[T,\phi^-] \ , \tag{4.14}$$

in contradiction with ϕ^- minimizing J in $\mathcal{P}_s$.

It is worth noticing that the calculations developed above also show that the system does not admit symmetric distribution functions $f_e\ne f^-$ merely extremizing the free energy. By a standard Euler-Lagrange argument, f_e should be indeed of the form (3.7), with its potential ϕ_e a solution in $\mathcal{P}_s$ of Eq. (3.9). However, nonsymmetric critical points are not excluded (although their existence seems quite unlikely).

C. Thermodynamic functions (Ω a disk)

We now compute the main thermodynamic functions associated to the equilibrium distribution function f^- (see also Refs. [5,8]). To simplify the expressions, we set

$$\theta := \frac{2}{\beta N} = \frac{2T}{N} = \frac{T}{T_c} \ , \quad V := |\Omega| = \pi R^2 \ . \qquad (4.15)$$

A straightforward calculation thus gives for $T > T_c$ $(\theta > 1)$

$$E_c^-(T,N) := E_c[f^-] = \tfrac{1}{2} N^2 \theta \ , \qquad (4.16)$$

$$E_p^-(T,V,N) := E_p[f^-] = \frac{N^2}{2}\left[\ln\left[\frac{V}{\pi}\right] + \theta + \theta^2 \ln\left[1 - \frac{1}{\theta}\right]\right] , \qquad (4.17)$$

$$E^-(T,V,N) := E[f^-] = \frac{N^2}{2}\left[\ln\left[\frac{V}{\pi}\right] + 2\theta + \theta^2 \ln\left[1 - \frac{1}{\theta}\right]\right] , \qquad (4.18)$$

$$S^-(T,V,N) := S[f^-] = N\left[\ln(\pi^2 e^3) + \ln\left[\frac{V}{\pi}\right] + \ln\theta + (2\theta - 1)\ln\left[1 - \frac{1}{\theta}\right]\right] , \qquad (4.19)$$

$$F^-(T,V,N) := F[T,f^-] = -\frac{N^2}{2}\left[(\theta-1)\ln\left[\frac{V}{\pi}\right] + \theta\ln(e\pi^2) + \theta\ln\theta + \theta(\theta-1)\ln\left[1 - \frac{1}{\theta}\right]\right] . \qquad (4.20)$$

The variations of E^-, S^-, and F^- as functions of θ are shown in Fig. 1. An important point to note is the monotonic increase of E^- and S^- from $-\infty$ up to $+\infty$ when θ varies from 1 to ∞. F^- increases from the finite value $-N^2 \ln(\pi\sqrt{e})$ (easily shown to be the infimum of F for $T = T_c$) up to some maximum value, and thus decreases to $-\infty$.

On the other hand, one gets for the "thermodynamic pressure" of the system [see also the virial theorem (4.28) below]

$$p_{\rm th}^-(T,V,N) := -\frac{\partial F^-}{\partial V} = \frac{N^2}{2V}(\theta - 1) = \frac{NT}{V}\left[1 - \frac{N}{2T}\right] = n^-(R)T = p^-(R) \qquad (4.21)$$

(where $n^- := n_{f^-}$ and $p^- := p_{f^-} := \frac{1}{2}\int f^- v^2 d\mathbf{v}$ is the thermal pressure) and for the density contrast

$$\eta^-(T,V,N) := \frac{n^-(0)}{n^-(R)} = \left[\frac{\theta}{\theta - 1}\right]^2 . \qquad (4.22)$$

As expected on intuitive grounds, the system is "kinetic energy" dominated at high temperature $T \gg T_c$, behaving almost like a perfect gas: $E^- \simeq E_c^- \simeq NT$, $p^- V \simeq NT$, and $\eta^- \simeq 1$ (quasiuniformity). At low temperature $T \gtrsim T_c$, on the contrary, the system is gravity dominated. When T decreases towards T_c, the inhomogeneity increases, more and more matter concentrating near the origin [$\eta^- \simeq (\theta-1)^{-2} \to \infty$, $p^- \to 0$, and $E^- \simeq E_p^- \simeq (N^2/2)\ln(\theta - 1) \to -\infty$].

D. System confined in an arbitrary domain

We now present a few remarks about the case where Ω is an arbitrary domain ($\Omega \neq \Omega^*$). We first note that for $T \geq T_c$, F is bounded from below by $F^-(T,|D|,N)$, with

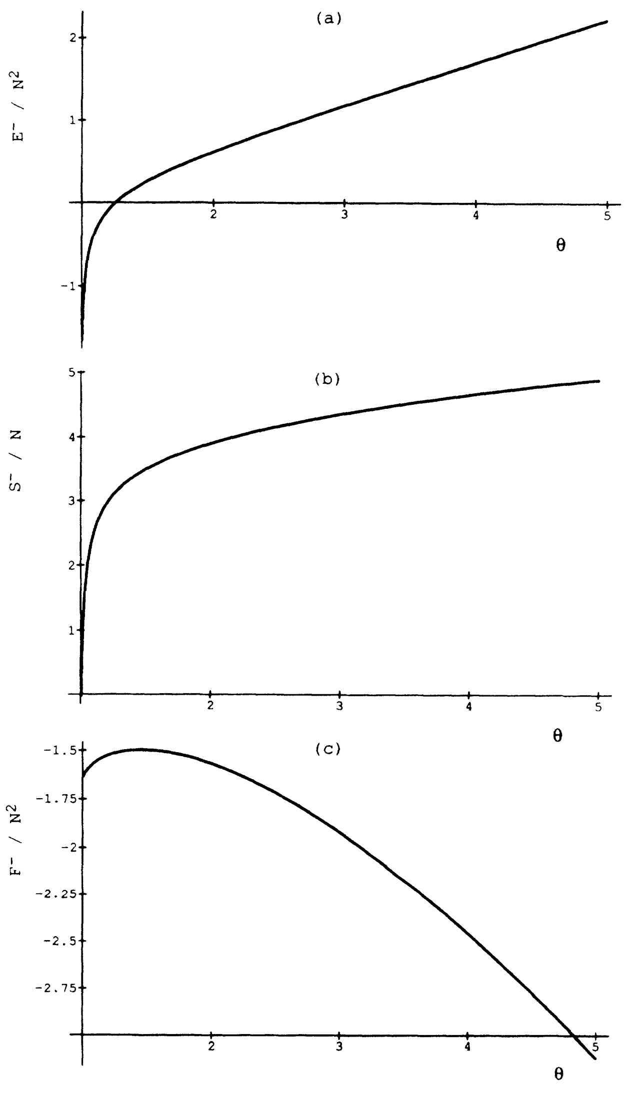

FIG. 1. (a) Energy E^-, (b) entropy S^-, and (c) free energy F^- for a system confined in a disk of unit radius ($V = \pi$) and in equilibrium with a thermal bath at temperature T, as a function of $\theta = T/T_c > 1$.

D a disk containing Ω ($\mathcal{F}[\Omega,N]\subset\mathcal{F}[D,N]$). A better lower bound is given by [15]

$$F^{-}(T,|\Omega|,N)\le F[T,f^{*}]\le F[T,f] , \quad (4.23)$$

where use has been made of the general property (3.5) of symmetrization and of the fact that we can apply the results of Secs. IV A–IV C to f^{*}, which belongs to $\mathcal{F}[\Omega^{*},N]$ [with Ω^{*} a disk; for $T=T_c$, F^{-} in Eq. (4.23) means $\lim_{T\to T_c^{+}} F^{-}$]. On the other hand, it is clear that F is not bounded from below on $\mathcal{F}$ for $T<T_c$ (we can apply the argument of Sec. IV A by choosing f_1 in $\mathcal{F}[D_1,N]$, with D_1 a disk contained in Ω). Therefore

$$F^{-}(T,\Omega,N):=\inf_{\mathcal{F}[\Omega,N]} F[T,f]\begin{cases}=-\infty & (T<T_c)\\ >F^{-}(T,|\Omega|,N)>-\infty & (T\ge T_c)\ .\end{cases} \quad (4.24)$$

The critical temperature T_c, below which the free energy is not bounded from below, does not depend on the shape and size of the box confining the system.

We shall not discuss here the existence of a minimizer f^{-} of F in $\mathcal{F}$ for $T>T_c$. We just note that, because of Eq. (3.13), f^{-} is necessarily of the form (3.7), with its potential ϕ^{-} a solution of the nonlinear integral equation (3.9) (with $\lambda=\beta$). In fact, it is easy to see that the Maxwellian factor in f^{-} solves the "velocity part" of the problem and that proving the existence of f^{-} amounts to proving the existence of a function n^{-} defined on Ω and minimizing the "macroscopic" free energy

$$F_0[T,n]:=E_{p0}[n]-TS_0[n]$$
$$:=-\int n(\mathbf{r})n(\mathbf{r}')\ln\frac{1}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}\,d\mathbf{r}'+T\int n\ln n\,d\mathbf{r} \quad (4.25)$$

in the set $\mathcal{N}[\Omega,N]$ of all the densities $n(\mathbf{r})\ge 0$ on Ω having the right N and finite "potential energy" and "entropy." Once the existence of a minimizer is known, computing it requires solving Eq. (3.8). This may be tried iteratively, the free energy thus decreasing at each step because of the properties of the transform: $f\to\tilde{f}_\beta$ [see Eq. (3.13)].

To conclude this brief study of the general case, we note the two following properties.

(a) Assume that F admits minimizers f_1^{-} and f_2^{-} at the temperatures T_1 and T_2, respectively, with $T_c<T_1<T_2$. Then obviously

$$E[f_1^{-}]-T_1S[f_1^{-}]\le E[f_2^{-}]-T_1S[f_2^{-}] , \quad (4.26a)$$

$$E[f_2^{-}]-T_2S[f_2^{-}]\le E[f_1^{-}]-T_2S[f_1^{-}] , \quad (4.26b)$$

which implies in particular (with obvious short notations)

$$0\le T_1(S_2^{-}-S_1^{-})\le E_2^{-}-E_1^{-}\le T_2(S_2^{-}-S_1^{-}) . \quad (4.27)$$

Then the energy and the entropy of f_2^{-} are at least as large as those of f_1^{-}.

(b) If f^{-} is a minimizer, we have by the virial theorem (Appendix C)

$$NT\left[1-\frac{N}{2T}\right]=NT\left[1-\frac{T_c}{T}\right]=2\int_{\partial\Omega}p^{-}(\mathbf{r}\cdot\hat{\mathbf{n}})ds , \quad (4.28)$$

where p^{-} is the thermal pressure of the gas defined in Sec. IV C. If Ω is "star shaped" (i.e, $\mathbf{r}\cdot\hat{\mathbf{n}}\ge 0$ on $\partial\Omega$, for some choice of the spatial origin O), Eq. (4.28) shows at once that no f^{-} can exist if $T\le T_c$. If f^{-} exists above T_c, then the pressure p^{-} must tend to zero on $\partial\Omega$ when $T\to T_c^{+}$, which, owing to the form of f^{-}, suggests the formation of a singularity as in the case where Ω is a disk. Actually, these arguments also hold true if we consider a mere critical point f_e of F, the virial theorem in the form (4.28) being valid for any isothermal equilibrium. Note that if we consider equilibria f_e which are relative minima of F, their nonexistence for $T\le T_c$ may be also proven by using the scaling argument presented at the end of Sec. IV A. We just need to take $f_1=f_e$ in Eqs. (4.10) and (4.11) and to note that, for Ω star shaped and $\alpha\ge 1$, we have $\Omega_\alpha:=\{\mathbf{r}|\alpha\mathbf{r}\in\Omega\}\subset\Omega$, whence f_α belongs to $\mathcal{F}$ too. Then Eq. (4.11) imposes $T>T_c$ if f_e is a relative minimum of the free energy. On the contrary, equilibria with $T<T_c$ can be easily constructed in special domains which have some "trapping regions." Consider, for instance, a domain Ω constituted of two disks of radius R connected by some thin tube. If the distance D between their centers is much larger than R, it is clear that we may have at a temperature T such that $T_c/2<T<T_c$, an equilibrium which coincides approximately in each of the disks with the exact symmetric equilibrium of the form (4.13) corresponding to $N/2$.

V. BOX-CONFINED ISOLATED SYSTEMS

We now assume that the system, still contained in the fixed domain Ω, is fully isolated, its energy thus keeping the fixed value E. As in the preceding section, we consider mainly the case where Ω is a disk.

A. Ω is a disk

We want to show that the set of admissible functions $\mathcal{F}[E]$ contains a unique entropy maximizer when Ω is a disk. In fact, this result follows easily from the conclusions of Sec. IV. Let us set indeed

$$f^{+}(E):=f^{-}(T(E)) , \quad (5.1)$$

where $f^{-}(T)$ [given by Eq. (4.13)] minimizes $F[T,\ ]$ in $\mathcal{F}$ and $T(E)>T_c$ is the unique solution of the equation

$$E[f^{-}(T(E))]=E \quad (5.2)$$

(that $T(E)$ exists and is unique for any value of E results at once from the monotonic increase of $E[f^{-}(T)]$ from $-\infty$ to $+\infty$ when T increases from T_c to $+\infty$). Then $f^{+}(E)$ maximizes the entropy in $\mathcal{F}[E]$, as we have for any f in that set

$$S[f^{+}(E)]-S[f]=\{F[T(E),f]-F[T(E),f^{-}(T(E))]\}\times[T(E)]^{-1}\ge 0 , \quad (5.3)$$

and our maximization problem has always at least one solution.

Let us now prove that $f^+(E)$ is the only solution. From the analysis of Sec. IV we have that $f^+(E)$ is the only function which (a) is cylindrically symmetric in $\mathbb{R}^2_r$, and (b) is of the form (4.6), with a potential solving Eq. (4.8) in $\mathcal{P}_s$. Then we just need to show that any entropy maximizer must satisfy both properties.

(a) Let us assume that there does exists a maximizer f_1^+, such that $(f_1^+)^* \neq f_1^+$. By the general properties of symmetrization, we have

$$S[(f_1^+)^*]=S[f_1^+] \,, \tag{5.4}$$

$$E^*:=E[(f_1^+)^*]<E[f_1^+] \,. \tag{5.5}$$

Using both relations and the fact that $S[f^+(E)]$ is an increasing function of E (S^- and E^- increasing monotonically with θ), we can write

$$S[f^+(E^*)]<S[f^+(E)]=S[f_1^+]=S[(f_1^+)^*] \,, \tag{5.6}$$

in contradiction with $f^+(E^*)$ maximizing the entropy in $\mathcal{F}[E^*]$. Therefore $f_1^+=(f_1^+)^*$ indeed.

(b) For a maximizer f_1^+, we have by Eq. (3.13)

$$\lambda\{E[(\tilde{f}_1^+)_\lambda]-E[f_1^+]\} \le S[(\tilde{f}_1^+)_\lambda]-S[f_1^+] \tag{5.7}$$

for any $\lambda>0$, with equality only if $f_1^+=(\tilde{f}_1^+)_\lambda$. Choosing λ in such a way that $E[(\tilde{f}_1^+)_\lambda]=E$, which is always possible as indicated at the end of Sec. III B, we are obviously in the equality case, and then $f_1^+=(\tilde{f}_1^+)_\lambda$ indeed (of course, the same result can be obtained by a standard Euler-Lagrange argument). Our uniqueness statement is thus established.

The equilibrium entropy (Fig. 2)

$$S^+(V,N,E):=S[f^+(E)] \tag{5.8}$$

has the parametric representation $S^+=S^-(\theta)$, $E=E^-(\theta)$ $(1<\theta<\infty)$, with $S^-(\theta)$ and $E^-(\theta)$ given by Eqs. (4.19) and (4.18), respectively. Elimination of θ cannot be done explicitly, but it is easy to check that the following inequalities are satisfied:

$$S^+(V,N,E) \le \frac{2E}{N}+N\ln(e\pi^2) \tag{5.9}$$

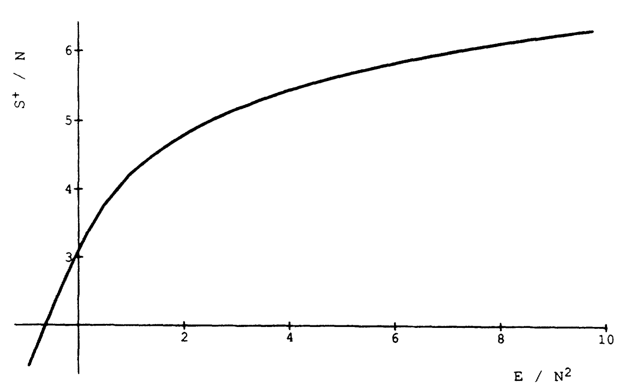

FIG. 2. Equilibrium entropy S^+ of an isolated system confined in a disk of unit radius ($V=\pi$), as a function of the energy E.

for E arbitrary and

$$S^+(V,N,E) \le N\ln\left[\frac{2\pi e^2 VE_r}{N^2}\right] \tag{5.10}$$

for

$$E_r:=E-N^2\ln(R/e) \ge N^2 \,, \tag{5.11}$$

S^+ behaving actually as the upper bounds (5.9) and (5.10) in the limits $E\to-\infty$ $(S^+\simeq 2E/N)$ and $E\to+\infty$ $(S^+\simeq N\ln E)$, respectively.

B. Ω is an arbitrary domain

When Ω is an arbitrary domain ($\Omega\neq\Omega^*$), we first note that entropy is still bounded from above on $\mathcal{F}[E]$ whichever be the value of E, i.e., $S^+(\Omega,N,E):=\sup_{\mathcal{F}[\Omega,N,E]} S[f]<\infty$. By the properties of symmetrization and the fact that $S^+(V,N,E)$ defined by Eq. (4.7) is a monotonically increasing function of E, we have indeed for any f of $\mathcal{F}[E]$ (with $V:=|\Omega^*|=|\Omega|$),

$$S[f]=S[f^*] \le S^+(V,N,E[f^*])<S^+(V,N,E) \,. \tag{5.12}$$

On the other hand, by arguments already used, it is easy to show that any entropy maximizer f^+—and more generally any critical point of the entropy in $\mathcal{F}[E]$—must be a Maxwell-Boltzmann function for some temperature T (not necessarily the same for different equilibria). When Ω is star shaped, the virial theorem (4.28) shows at once that $T>T_c$. That this inequality holds necessarily for (possibly relative) maximum entropy states may also be seen by adapting the scaling argument of Sec. IV A. f_e being an equilibrium distribution at temperature T, set for any $\alpha\ge 1$ and $\gamma>0$,

$$f_{\alpha\gamma}(\mathbf{r},\mathbf{v})=\alpha^2\gamma^2 f_e(\alpha\mathbf{r},\gamma\mathbf{v}) \,. \tag{5.13}$$

Clearly $f_{\alpha\gamma}$ belongs to $\mathcal{F}$ (as already noted, for Ω star shaped and $\alpha\ge 1$, $\Omega_\alpha:=\{\mathbf{r}|\alpha\mathbf{r}\in\Omega\}\subset\Omega$). Moreover, if we impose

$$N\ln\alpha=\left[\frac{1}{\gamma^2}-1\right]\frac{E_c[f_e]}{N}=T\left[\frac{1}{\gamma^2}-1\right] \tag{5.14}$$

(whence $\gamma\le 1$), $f_{\alpha\gamma}$ belongs to $\mathcal{F}[E]$ too and

$$S[f_{\alpha\gamma}]=S[f_e]-N\ln\alpha^2\gamma^2 =S[f_e]-N\left[\frac{T}{T_c}\left[\frac{1}{\gamma^2}-1\right]+\ln\gamma^2\right] \,. \tag{5.15}$$

As we must have $S[f_{\alpha\gamma}]<S[f_e]$ for $\gamma<1$, Eq. (5.15) imposes $T>T_c$ indeed. On the contrary, equilibria with $T<T_c$ may exist in some non-star-shaped domain. This results immediately from the analysis at the end of Sec. IV D and the fact that any local free energy minimizer is also a local entropy maximizer (see the argument at the beginning of Sec. V A).

The problem of the existence of a maximizer f^+ will not be discussed here. We just remark that it amounts to prove that the functional

$$S_1[n,E]:=\begin{cases} S_0[n]+N\ln\left[\frac{2\pi e}{N}(E-E_{p0}[n])\right], & E>E_{p0}[n] \\ -\infty, \quad E\le E_{p0}[n] \end{cases} \tag{5.16}$$

has a maximizer in $\mathcal{N}[\Omega,N]$ (with the notations of Sec. IV D), the "velocity part" of the problem being solved here too by the Maxwellian distribution.

VI. SYSTEMS CONFINED BY AN EXTERNAL PRESSURE

The case where the system is confined by an external constant pressure P can also be dealt easily by using the results of Secs. IV and V. Consider first the system in contact with a thermal bath at temperature $T=\beta^{-1}$, a stable equilibrium being thus a global minimizer of the free enthalpy $G[P,T,\ ]$ in $\mathcal{G}$. By using once more symmetrization, we see immediately that, for any f in $\mathcal{G}$, we have

$$G[P,T,f^*]\le G[P,T,f] \tag{6.1}$$

($|\Omega_{f^*}|=|\Omega_f|$), and then we can consider our minimization problem in the subset $\mathcal{G}_s$ of $\mathcal{G}$ containing all the distribution functions which are cylindrically symmetric in $\mathbb{R}_r^2$, with Ω_f a disk of center O and radius R_f. Moreover, when $T>T_c$, it is quite clear that we have for any function f of $\mathcal{G}_s$ and for some function L (see Eq. (4.20))

$$G[P,T,f]\ge P|\Omega_f|+F^-(T,|\Omega_f|,N) = P|\Omega_f|-NT\left[1-\frac{N}{2T}\right]\ln|\Omega_f|+L(N,T)\,, \tag{6.2}$$

with equality if and only if f is the minimizer $f^-(T,|\Omega_f|,N)$ of $F[T,\ ]$ in $\mathcal{F}[\Omega_f,N]$. As the right-hand side of Eq. (6.2) takes its minimum value for

$$|\Omega_f|=V^-(P,T,N):=\frac{NT}{P}\left[1-\frac{N}{2T}\right], \tag{6.3}$$

we can thus conclude that the free enthalpy is uniquely minimized in $\mathcal{G}$ by the function

$$g^-(P,T,N)=f^-(T,V^-(P,T,N),N)\,. \tag{6.4}$$

Note that Eq. (6.3) can also be written $p^-(T,V^-,N)=P$: As expected *a priori*, there is pressure balance at the boundary of the system when equilibrium holds. Also, the volume of the system increases from 0 to $+\infty$ when T increases from T_c to $+\infty$. From Eqs. (4.20) and (6.3), we have

$$\begin{aligned} G^-(P,T,N)&:=G[P,T,g^-] \\ &=-\frac{N^2}{2}\left\{1+2\theta\ln\pi+\theta\ln\theta+(\theta-1)\ln\left[\frac{N^2}{2\pi P}(\theta-1)\right]+\theta(\theta-1)\ln\left[\frac{\theta-1}{\theta}\right]\right\} \\ &\ge G^-(P,T_c,N)=-\frac{N^2}{2}\ln(e\pi^2)\,. \end{aligned} \tag{6.5}$$

The variation of G^- with T is shown in Fig. 3. On the other hand, it is also obvious that G is not bounded from below for $T<T_c$ and that it is bounded from below, but does not reach its infimum, when $T=T_c$.

Let us now consider the case where the system is

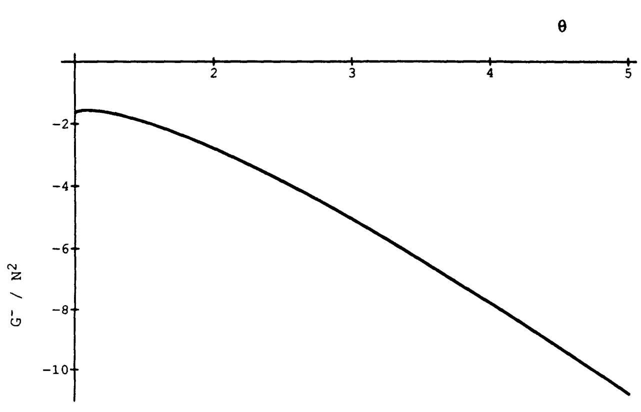

FIG. 3. Free enthalpy G^- of a system confined by the external pressure $P=N^2/2\pi$ and in equilibrium with a thermal bath at temperature T, as a function of $\theta=T/T_c>1$.

thermally insulated and thus evolves at constant enthalpy H. By arguments similar to the ones which have allowed us to pass from the function f^- minimizing the free energy in $\mathcal{F}$ to the function f^+ maximizing the entropy in $\mathcal{F}[E]$, it is straightforward to show that the entropy is always uniquely maximized in $\mathcal{G}[H]$ by the symmetric function

$$g^+(H,N,P)=g^-(P,T(H,P,N),N)\,, \tag{6.6}$$

with g^- given by Eq. (6.4) and $T(H,P,N)$ the unique solution of

$$H=\frac{N^2}{2}\left\{3\theta-1+\theta^2\ln\left[1-\frac{1}{\theta}\right]+\ln\left[\frac{N^2}{2\pi P}(\theta-1)\right]\right\} \tag{6.7}$$

(that this equation for $\theta=T/T_c$ has a unique solution in $]1,\infty[$ for an arbitrary value of H results at once from the fact that its right-hand side increases monotonically from $-\infty$ to $+\infty$ when θ increases from 1 to $+\infty$). The value of the entropy $S^+(H,P,N):=S[g^+]$ at equilibrium is shown in Fig. 4. The volume of the system is given by

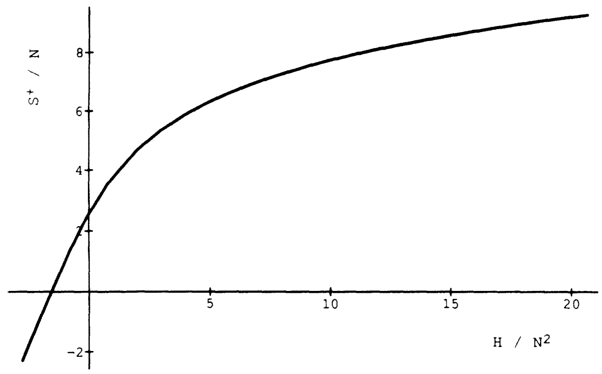

FIG. 4. Equilibrium entropy S^+ of a system confined by the external pressure $P=N^2/2\pi$ and thermally insulated, as a function of the enthalpy H.

$$V^+(H,P,N)=V^-(P,T(H,P,N),N) \ . \tag{6.8}$$

It increases from 0 to $+\infty$ when H increases from $-\infty$ to $+\infty$.

VII. CONCLUSION

By making an essential use of symmetrization arguments and of the Moser-Trudinger inequality, we have shown in this paper the existence and uniqueness of a globally stable mean field thermodynamic equilibrium for a 2D system constituted of N particles interacting gravitationally and (i) confined either by a rigid circular box or by an external pressure, and (ii) either maintained in contact with a thermal bath at temperature $T>T_c=N/2$, or thermally insulated. In any case, the equilibrium corresponds to a cylindrically symmetric Maxwell-Boltzmann distribution function—the free boundary in the case of a pressure-confined system adopting spontaneously a circular shape. The equilibrium exhibits a central condensation which becomes more and more pronounced when one considers (depending on the situations) either temperature approaching T_c or either energy or enthalpy taking lower and lower values. For a system in contact with a thermostat at temperature $T\le T_c$, no regular equilibria (with finite energy and entropy) exist and the natural state seems to correspond to the whole system being concentrated at one point of the domain [1,5,9]. We have also considered the case of a system confined inside a rigid box Ω of arbitrary shape and shown that, in that situation too, the free energy is bounded from below if and only if $T\ge T_c$ (system in contact with a heat bath), while the entropy is bounded from above whichever the value of the energy (isolated system). We have also proven that, if Ω is star shaped, there does not exist any equilibrium (stable or unstable) having a temperature $T\le T_c$.

The qualitative behavior of a 2D system thus appears to differ quite substantially from that of a 3D system. In the framework of the phenomenological mean field approximation, it has been proven indeed by Antonov [19] (who used earlier work by Emden [20]) and thus by many other authors (e.g., [2,21,22]), that there is no smooth global maximum entropy state for an isolated object of mass M and energy E confined inside a sphere of radius R. But for a large enough value of the dimensionless parameter $\Lambda:=RE/G_NM^2$ ($\Lambda>-0.335$), there is an available isothermal equilibrium which is a local maximum of the entropy if the density contrast $n(0)/n(R)<709$ and a saddle point of the entropy if $n(0)/n(R)>709$—in which case one gets an unstable situation known as the *gravothermal catastrophe.*

It should be noted, however, that these results have been obtained in a classical context. The situation is different in the framework of nonrelativistic quantum mechanics, where a system of N gravitationally interacting fermions can be shown to have a ground state of finite energy, the latter scaling like $-N^{7/3}$ [23]. By considering the limit $N\to\infty$ with an appropriate scaling of some physical quantities, Thirring and collaborators [24] were then able to prove the existence for such a system of a mean field state, characterized by a Fermi-Dirac density self-consistently coupled to the mean gravitational potential by the Poisson equation. Like in the 2D classical case considered in this paper, a maximum entropy state does now exist and the standard 3D gravothermal catastrophe appears to be avoided—in fact as a consequence of Fermi exclusion principle. But gravothermal catastrophe turns out to be actually recovered if the mass of the system exceeds a critical value (the so-called "Chandrasekhar mass," of the order of a few solar masses). In that case, the velocities of most of the particles in the ground state computed in Thirring's model exceed the speed of light. Relativistic effects can no longer be neglected, and taking them into account results into the nonexistence of an equilibrium (see, e.g., Ref. [25] for a review).

ACKNOWLEDGMENTS

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APPENDIX A: THE LOGARITHMIC POTENTIAL

Let $n(\mathbf{r})$ be a density of finite total mass N concentrated inside the bounded domain Ω of the plane ($n=0$ in $\mathbb{R}_r^2/\bar{\Omega}$). Then its associated logarithmic potential [26,27] in $\mathbb{R}_r^2$ is given by

$$\phi(\mathbf{r})=-2\int_\Omega n(\mathbf{r}')\ln\frac{1}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}' \ . \tag{A1}$$

Asymptotically, one has

$$\phi(\mathbf{r}) \underset{r\to\infty}{=} 2N\ln r+O(r^{-1}) \ , \tag{A2}$$

$$\nabla\phi(\mathbf{r}) \underset{r\to\infty}{=} \frac{2N}{r}\hat{\mathbf{r}}+O(r^{-2}) \ . \tag{A3}$$

Let n_1 and n_2 be two densities on Ω with the same N, and ϕ_1 and ϕ_2 be, respectively, the potentials they create. Applying Gauss's theorem in a large disk D_R of radius R centered at the origin and containing Ω, we get

$$\int_\Omega (n_1\phi_2-n_2\phi_1)d\mathbf{r}=\frac{1}{4\pi}\int_{D_R}(\phi_2\Delta\phi_1-\phi_1\Delta\phi_2)d\mathbf{r}$$
$$=\frac{1}{4\pi}\int_{\partial D_R}\left[\phi_2\frac{\partial\phi_1}{\partial r}-\phi_1\frac{\partial\phi_2}{\partial r}\right]ds\ , \tag{A4}$$

whence, making $R\to\infty$ and using Eqs. (A2) and (A3),

$$\int_\Omega n_1\phi_2 d\mathbf{r}=\int_\Omega n_2\phi_1 d\mathbf{r} \tag{A5}$$

(the "reciprocity theorem"). Similarly

$$-\int_\Omega (n_2-n_1)(\phi_2-\phi_1)d\mathbf{r}$$
$$=\frac{1}{4\pi}\int_{D_R}|\nabla(\phi_2-\phi_1)|^2 d\mathbf{r}$$
$$-\frac{1}{4\pi}\int_{D_R}(\phi_2-\phi_1)\frac{\partial}{\partial r}(\phi_2-\phi_1)ds$$
$$\underset{R\to\infty}{=}\frac{1}{4\pi}\int|\nabla(\phi_2-\phi_1)|^2 d\mathbf{r}\geq 0\ , \tag{A6}$$

with equality holding only if $n_1=n_2$ (the "positivity theorem").

In the case where Ω is a disk of center 0 and radius R, and $n(\mathbf{r})=n(r)$, we have by Gauss's theorem

$$2\pi r\frac{d\phi}{dr}=4\pi\int_0^r(2\pi s)n(s)ds=:4\pi N(r) \tag{A7}$$

and

$$\phi(r)=\begin{cases}-2\int_r^R\frac{N(s)}{s}ds+2N\ln R\ , & r\leq R\\ 2N\ln r\ , \quad R\leq r\ ,\end{cases} \tag{A8}$$

where use has been made of Eq. (A2). On the boundary $\partial\Omega$, we have

$$\phi(R)=2N\ln R\ , \tag{A9}$$

$$\frac{d\phi}{dr}(R)=\frac{2N}{R}\ . \tag{A10}$$

Note that, in any case, the cylindrically symmetric potential ϕ is continuously differentiable but may be at the origin.

APPENDIX B: MATHEMATICAL RESULTS

We first recall two classical inequalities which play a crucial role in this paper: The Riesz inequality (in Lieb's strong form) and the Moser-Trudinger inequality. Thus we give a proof of a result used in Sec. IV B.

1. Riesz's inequality in $\mathbb{R}_r^2$

Theorem [16]. Let v be a positive spherically symmetric strictly decreasing function of $\mathbb{R}_r^2$. For any two non-negative functions $u\in L^p(\mathbb{R}_r^2)$ and $w\in L^q(\mathbb{R}_r^2)(p^{-1}+q^{-1}=1)$, we have

$$\int u(\mathbf{r})v(|\mathbf{r}-\mathbf{r}'|)w(\mathbf{r}')d\mathbf{r}\,d\mathbf{r}'$$
$$\leq\int u^*(r)v(|\mathbf{r}-\mathbf{r}'|)w^*(r')d\mathbf{r}\,d\mathbf{r}'\ , \tag{B1}$$

with equality holding if and only if $u\equiv u^*$ and $w\equiv w^*$ up to a translation (of course, u^* and w^* are, respectively, the rearrangements of u and w defined in Sec. III A). ■

Let us now assume that $u\geq 0$ has a compact support Ω of diameter D and set $v(t)=\ln(2D/t)$ for $0<t<D$ and $v(t)$ equal to some arbitrary function decreasing from ln2 to 0 for $D<t<\infty$. Then we have by the previous theorem (with $v=u$)

$$\int_{\Omega\times\Omega}u(\mathbf{r})u(\mathbf{r}')\ln\frac{2D}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}\,d\mathbf{r}'$$
$$\leq\int_{\Omega^*\times\Omega^*}u^*(r)u^*(r')\ln\frac{2D}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}\,d\mathbf{r}'\ , \tag{B2}$$

whence (because $\int_\Omega u\,d\mathbf{r}=\int_{\Omega^*}u^*d\mathbf{r}$)

$$\int u(\mathbf{r})u(\mathbf{r}')\ln\frac{1}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}\,d\mathbf{r}'$$
$$\leq\int u^*(r)u^*(r')\ln\frac{1}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}\,d\mathbf{r}'\ , \tag{B3}$$

with equality holding if and only if $u\equiv u^*$ up to a translation.

2. Moser and Trudinger's inequality in $\mathbb{R}_r^2$

Theorem [17,28]. Let Ω be a bounded domain of $\mathbb{R}_r^2$ and u be a function of the Sobolev space $H_0^1(\Omega)$ (i.e., vanishing on $\partial\Omega$) satisfying

$$\int_\Omega|\nabla u|^2 d\mathbf{r}\leq 1\ . \tag{B4}$$

Then we have

$$\int_\Omega e^{4\pi u^2}d\mathbf{r}\leq c|\Omega|\ , \tag{B5}$$

where c is a constant (independent of Ω). ■

As a simple corollary of this theorem, we have for any function $v\in H_0^1(\Omega)$

$$\ln\int_\Omega e^{\delta v}d\mathbf{r}\leq\ln(c|\Omega|)+\frac{\delta^2}{16\pi}\int_\Omega|\nabla v|^2 d\mathbf{r}\ , \tag{B6}$$

where δ is an arbitrary constant. To prove this result, we define a function u fulfilling the conditions of the theorem by

$$v=:u\left[\int_\Omega|\nabla v|^2 d\mathbf{r}\right]^{1/2}\ , \tag{B7}$$

and we note that v satisfies the elementary inequality [28]

$$\delta v\leq 4\pi u^2+\frac{\delta^2}{16\pi}\left[\int_\Omega|\nabla v|^2 d\mathbf{r}\right]\ . \tag{B8}$$

Then we exponentiate and integrate over Ω both members of Eq. (B8) and we apply Eq. (B5).

On the other hand, we have for any two functions u and v belonging to $H_0^1(\Omega)$

$$\left|\int_\Omega(e^v-e^u)d\mathbf{r}\right|$$
$$\leq(c|\Omega|)^{1/2}\left[\int_\Omega|v-u|^2 d\mathbf{r}\right]^{1/2}$$
$$\times\exp\left[\frac{1}{4\pi}\int_\Omega|\nabla v|^2 d\mathbf{r}+\frac{1}{4\pi}\int_\Omega|\nabla u|^2 d\mathbf{r}\right]\ . \tag{B9}$$

This relation is proven by using the elementary inequality

$$|e^{v}-e^{u}|\le|v-u|e^{|v|+|u|} \tag{B10}$$

and twice Schwartz's inequality and Eq. (B6) above.

3. Proof of the existence of a minimizer of $J[T,\phi]$

We first prove the following result (which has already appeared in the literature [6,13], but only with very sketchy proofs

Theorem. Let Ω be a bounded domain of $\mathbb{R}_r^2$. Then the functional

$$J_0[v]:=\frac{1}{8\pi}\int_\Omega|\nabla v|^2 d\mathbf{r}-NT\ln\left[\int_\Omega e^{-\beta v}d\mathbf{r}\right] \tag{B11}$$

admits a minimizer in $H_0^1(\Omega)$ when $\beta^{-1}=T>T_c:=N/2$. ∎

Proof. By Eq. (B6), we have for $T>T_c$

$$J_0[v]\ge\frac{1}{8\pi}\left[1-\frac{T_c}{T}\right]\int_\Omega|\nabla v|^2 d\mathbf{r}-NT\ln(c|\Omega|)$$
$$\ge -NT\ln(c|\Omega|)\,, \tag{B12}$$

and $J_0[v]$ has a finite largest lower bound $J_0^-:=\inf_{H_0^1(\Omega)}J_0[v]$. Then consider a minimizing sequence $\{v_k\}$ in $H_0^1(\Omega)$ (i.e., a sequence such that $\lim_{k\to\infty}J_0[v_k]=J_0^-$), with $J_0^-\le J_0[v_k]\le J_0^-+1$, say. By Eq. (B12), $\{v_k\}$ is uniformly bounded in $H_0^1(\Omega)$. Then a subsequence of it (still denoted as $\{v_k\}$) converges weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$ to a function v^- of $H_0^1(\Omega)$ [17]. Of course $J_0[v^-]\ge J_0^-$ while, on the other hand, $J_0[v^-]\le J_0^-$ as

$$J_0[v^-]-J_0^-\le\frac{1}{8\pi}\left[\int_\Omega|\nabla v^-|^2 d\mathbf{r}-\varliminf_{k\to\infty}\int_\Omega|\nabla v_k|^2 d\mathbf{r}\right]$$
$$+\varlimsup_{k\to\infty}NT\ln\left[1+\left[\int_\Omega e^{-\beta v_k}d\mathbf{r}-\int_\Omega e^{-\beta v^-}d\mathbf{r}\right]\left[\int_\Omega e^{-\beta v^-}d\mathbf{r}\right]^{-1}\right]\le 0\,, \tag{B13}$$

the first term in the second member of Eq. (B13) being nonpositive as a consequence of the weak lower semicontinuity of the first term on the right-hand side of Eq. (B12) [17], while the second one vanishes owing to Eq. (B9), the convergence of v_k to v^- in $L^2(\Omega)$, and the boundedness of v_k in $H_0^1(\Omega)$. Thus $J_0[v^-]=J_0^-$ and v^- is a minimizer of J. ∎

Consider now the case where Ω is a disk. Then, by a symmetrization argument, we can conclude that there is a cylindrically symmetric minimizer v^-. On the other hand, the function $\Phi^-:=v^-+2N\ln R$ is clearly a minimizer of J in $\mathcal{P}_s$, as J defined on $2N\ln R+H_0^1(\Omega)$ and J_0 defined on $H_0^1(\Omega)$ differ only by constant terms.

APPENDIX C: THE VIRIAL THEOREM

The virial theorem for 3D systems is a well known and widely used integral equality [29]. Here we derive a form of this theorem applying to 2D equilibria occupying some domain Ω. Although our result is likely to be known, we have not found any reference to it in the literature. For simplicity, we consider an equilibrium described by a distribution function f_e which is a function of the only energy of a particle [i.e., $f_e=f_e(v^2/2+\phi_e)$, with $\phi_e:=\phi_{f_e}$]. For such an equilibrium, we have obviously

$$\nabla p_e=-n_e\nabla\phi_e=-\frac{1}{4\pi}\nabla\cdot\left\{\nabla\phi_e\otimes\nabla\phi_e-\frac{|\nabla\phi_e|^2}{2}\mathsf{I}\right\}\,, \tag{C1}$$

where $\mathsf{I}:=[\delta_i^j]$

$$p_e=\tfrac{1}{2}\int v^2 f_e d\mathbf{v} \tag{C2}$$

is the thermal pressure of the particles, and use has been made of Poisson's equation (2.5a) to get the last equality. Multiplying Eq. (C1) by $\mathbf{r}$, integrating the result over Ω and applying Gauss's theorem, we obtain eventually

$$2\int_\Omega p_e d\mathbf{r}=\int_{\partial\Omega}p_e\mathbf{r}\cdot\hat{\mathbf{n}}\,ds+\frac{1}{4\pi}\int_{\partial\Omega}\left\{(\mathbf{r}\cdot\nabla\phi_e)(\hat{\mathbf{n}}\cdot\nabla\phi_e)-\frac{|\nabla\phi_e|^2}{2}(\mathbf{r}\cdot\hat{\mathbf{n}})\right\}ds\,, \tag{C3}$$

where $\hat{\mathbf{n}}$ is the external normal to $\partial\Omega$.

Let us now consider a disk D_R of center 0 and of radius large enough for D_R containing Ω. In D_R/Ω, the vacuum gravitational field still satisfies Eq. (C1) (with $p_e=n_e=0$) and thus we get, by following once more the procedure leading to Eq. (C3),

$$\int_{\partial\Omega}\left\{(\mathbf{r}\cdot\nabla\phi_e)(\hat{\mathbf{n}}\cdot\nabla\phi_e)-\frac{|\nabla\phi_e|^2}{2}(\mathbf{r}\cdot\hat{\mathbf{n}})\right\}ds$$
$$=\frac{R^2}{2}\int_0^{2\pi}\left[\left|\frac{\partial\phi_e}{\partial r}\right|^2-\frac{1}{R^2}\left|\frac{\partial\phi_e}{\partial\theta}\right|^2\right](R,\theta)d\theta$$
$$\underset{R\to\infty}{=}4\pi N^2\,, \tag{C4}$$

where (r,θ) denote polar coordinates and the last equality has been obtained by using the asymptotic expression for $\nabla\phi_e$, which guarantees that $\lim_{R\to\infty}|R\,\partial\phi_e/\partial r|=2N$ while $\lim_{R\to\infty}|\partial\phi_e/\partial\theta|=0$ (Appendix A).

Combining Eqs. (C3) and (C4), we obtain the sought expression

$$2\int_\Omega p_e d\mathbf{r}=N^2+\int_{\partial\Omega}p_e\mathbf{r}\cdot\hat{\mathbf{n}}\,ds\,. \tag{C5}$$

In the case where the equilibrium corresponds to a Maxwell-Boltzmann distribution function with a temperature T, we have $p_e=n_eT$ and Eq. (C5) gives

$$NT\left[1-\frac{\beta N}{2}\right]=NT\left[1-\frac{T_c}{T}\right]=\tfrac{1}{2}T\int_{\partial\Omega}n_e\mathbf{r}\cdot\hat{\mathbf{n}}\,ds\,. \tag{C6}$$

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